Periodicity of Adams operations on the Green ring of a finite group

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December 15, 2009

Abstract

The Adams operations ψ_Λ^n and ψ_S^n on the Green ring of a group G over a field K provide a framework for the study of the exterior powers and symmetric powers of KG-modules. When G is finite and K has prime characteristic p we show that ψ_Λ^n and ψ_S^n are periodic in n if and only if the Sylow p-subgroups of G are cyclic. In the case where G is a cyclic p-group we find the minimum periods and use recent work of Symonds to express ψ_S^n in terms of ψ_Λ^n.

1 Introduction

In this paper we study the Adams operations defined from the exterior powers $\Lambda^n(V)$ and the symmetric powers $S^n(V)$ of a KG-module V, where G is a group and K is a field. By a KG-module we shall always mean a finite-dimensional right KG-module. The modules $\Lambda^n(V)$ and $S^n(V)$ are of fundamental importance in the study of KG-modules. Thus, for a given V, we would like to know the structure of $\Lambda^n(V)$ and $S^n(V)$. For example, we would like to be able to express these modules up to isomorphism as direct sums of indecomposables.

When describing modules up to isomorphism it is useful to work in the Green ring (or representation ring) R_{KG}. This has a $\mathbb{Z}$-basis consisting of representatives of the isomorphism classes of indecomposable KG-modules, and multiplication comes from tensor products. Each KG-module may be regarded, up to isomorphism, as an element of R_{KG}.

For each $n > 0$ there are $\mathbb{Z}$-linear maps ψ_Λ^n and ψ_S^n from R_{KG} to R_{KG}, called the Adams operations (see Section 2). The module $\Lambda^n(V)$ may be expressed, in $\mathbb{Q} \otimes_{\mathbb{Z}} R_{KG}$, as a polynomial in $\psi_\Lambda^1(V), \ldots, \psi_\Lambda^n(V)$. Similarly, $S^n(V)$ is a polynomial

in $\psi_S^1(V), \ldots, \psi_S^n(V)$. Thus $\Lambda^n(V)$ and $S^n(V)$ are determined, up to isomorphism, by knowledge of the Adams operations.

The advantage of working with the Adams operations is that they have some rather simple properties (see Section 2). For example, when n is not divisible by the characteristic of K, we have $\psi_\Lambda^n = \psi_S^n$ and ψ_Λ^n is a ring endomorphism of R_{KG}. Furthermore, according to the evidence available in [5] and in some of the references cited there, expressions for $\psi_\Lambda^n(V)$ and $\psi_S^n(V)$ within R_{KG} are often much simpler in form than expressions for $\Lambda^n(V)$ and $S^n(V)$.

In a previous paper [5] we studied the Adams operations for a cyclic p-group C in prime characteristic p. We showed how to calculate ψ_Λ^n for all n not divisible by p (when we also have $\psi_\Lambda^n = \psi_S^n$). Here we shall continue our study of ψ_Λ^n and ψ_S^n on R_{KC}, allowing n to be an arbitrary positive integer. In Sections 4 and 6 we shall establish periodicity in n: explicitly, $\psi_\Lambda^n = \psi_\Lambda^{n+2q}$ and $\psi_S^n = \psi_S^{n+2q}$, where $q = |C|$. The proofs of these results make use of our earlier work [5], and the proof for ψ_S^n also relies upon deep work of Symonds [8] that gives periodicity 'modulo induced modules' of the symmetric powers of indecomposable KC-modules. Symonds [8] also obtained a result that expresses symmetric powers of indecomposable KC-modules in terms of exterior powers, but again 'modulo induced modules'. In Section 6 we show that there is a corresponding result that expresses ψ_S^n directly in terms of ψ_Λ^n.

The other main results of this paper concern the periodicity of ψ_Λ^n and ψ_S^n for an arbitrary finite group G in prime characteristic p. We show in Section 3 that ψ_Λ^n is periodic in n if and only if the Sylow p-subgroups of G are cyclic. In Section 5 we obtain a similar result for ψ_S^n, but the proof is much harder and makes use of [8]. In Section 7 we obtain some lower bounds for the minimum periods.

2 Preliminaries

Let G be a group and K a field. We consider (finite-dimensional right) KG-modules and denote the associated Green ring by R_{KG}, as in Section 1. For any KG-module V, we also write V for the corresponding element of R_{KG}. Thus, for KG-modules V and W, we have $V = W$ in R_{KG} if and only if $V \cong W$. The elements $V + W$ and VW of R_{KG} correspond to the modules $V \oplus W$ and $V \otimes_K W$, respectively. The identity element of R_{KG} is the one-dimensional KG-module on

which G acts trivially, usually denoted by 1 when regarded as an element of R_{KG} and K when regarded as a KG-module.

If V is a KG-module and n is a non-negative integer, then we regard the nth exterior power $\Lambda^n(V)$ and the nth symmetric power $S^n(V)$ as elements of R_{KG}. In particular, $\Lambda^0(V) = S^0(V) = 1$ and $\Lambda^1(V) = S^1(V) = V$. If V has dimension r then $\dim \Lambda^n(V) = \binom{r}{n}$, where for $n > r$ we define $\binom{r}{n} = 0$. Thus we see that $\Lambda^r(V)$ is a one-dimensional module, whilst $\Lambda^n(V) = 0$ for all $n > r$.

For any KG-module V we define elements of the power-series ring $R_{KG}[[t]]$ by

$$\Lambda(V,t) = 1 + \Lambda^1(V)t + \Lambda^2(V)t^2 + \cdots ,$$

$$S(V,t) = 1 + S^1(V)t + S^2(V)t^2 + \cdots .$$

(In fact $\Lambda(V,t)$ belongs to the polynomial ring $R_{KG}[t]$.) We extend R_{KG} to a ring $\mathbb{Q}R_{KG}$, allowing coefficients from $\mathbb{Q}$, so that $\mathbb{Q}R_{KG} \cong \mathbb{Q} \otimes_{\mathbb{Z}} R_{KG}$. The formal expansion of $\log(1+x)$ yields elements $\log \Lambda(V,t)$ and $\log S(V,t)$ of $\mathbb{Q}R_{KG}[[t]]$. Thus for $n > 0$ we may define elements $\psi^n_\Lambda(V)$ and $\psi^n_S(V)$ of $\mathbb{Q}R_{KG}$ by the equations

$$\psi^1_\Lambda(V) - \psi^2_\Lambda(V)t + \psi^3_\Lambda(V)t^2 - \cdots = \frac{d}{dt}\log \Lambda(V,t) = \Lambda(V,t)^{-1}\frac{d}{dt}\Lambda(V,t), \tag{2.1}$$

$$\psi^1_S(V) + \psi^2_S(V)t + \psi^3_S(V)t^2 + \cdots = \frac{d}{dt}\log S(V,t) = S(V,t)^{-1}\frac{d}{dt}S(V,t). \tag{2.2}$$

It is easily verified that $\psi^1_\Lambda(V) = \psi^1_S(V) = V$ for all KG-modules V. In fact, it can be shown that $\psi^n_\Lambda(V), \psi^n_S(V) \in R_{KG}$ for all $n > 0$. For the trivial module K we have $\psi^n_\Lambda(K) = \psi^n_S(K) = K$ for all $n > 0$. Furthermore,

$$\psi^n_\Lambda(V+W) = \psi^n_\Lambda(V) + \psi^n_\Lambda(W), \quad \psi^n_S(V+W) = \psi^n_S(V) + \psi^n_S(W),$$

for all $n > 0$ and all KG-modules V and W (see [3] for details). Thus the definitions of ψ^n_Λ and ψ^n_S may be extended to give $\mathbb{Z}$-linear maps

$$\psi^n_\Lambda : R_{KG} \to R_{KG}, \quad \psi^n_S : R_{KG} \to R_{KG},$$

called the nth Adams operations on R_{KG}. We write δ for the 'dimension' map $\delta : R_{KG} \to \mathbb{Z}$. This is the ring homomorphism satisfying $\delta(V) = \dim V$ for every KG-module V. Since there is an embedding $\nu : \mathbb{Z} \to R_{KG}$ given by $\nu(k) = k.1$ for all $k \in \mathbb{Z}$ we may also regard δ as an endomorphism of R_{KG}. By [5, (2.2)] we have

$$\delta(\psi^n_\Lambda(A)) = \delta(\psi^n_S(A)) = \delta(A), \tag{2.3}$$

for all $A \in R_{KG}$ and all $n > 0$.

We now extend R_{KG} to a ring $\mathbb{C}R_{KG}$, allowing coefficients from $\mathbb{C}$, so that $\mathbb{C}R_{KG} \cong \mathbb{C} \otimes_{\mathbb{Z}} R_{KG}$ and $\mathbb{C}R_{KG}$ is a $\mathbb{C}$-algebra. The Adams operations on R_{KG} extend to $\mathbb{C}$-linear maps ψ^n_Λ and ψ^n_S on $\mathbb{C}R_{KG}$. The following result may be found in [3, Theorem 5.4 and (4.4)].

Proposition 2.1. *Let G be a group, K a field, and n a positive integer not divisible by the characteristic of K. Then $\psi^n_\Lambda = \psi^n_S$ and ψ^n_Λ is a ring endomorphism of R_{KG} and an algebra endomorphism of $\mathbb{C}R_{KG}$. If V is a KG-module then $\psi^n_\Lambda(V)$ is a $\mathbb{Z}$-linear combination of direct summands of $V^{\otimes n}$. Furthermore, under composition of maps we have $\psi^n_\Lambda \circ \psi^{n'}_\Lambda = \psi^{nn'}_\Lambda$ and $\psi^n_S \circ \psi^{n'}_S = \psi^{nn'}_S$ for every positive integer n'.*

For any KG-module V, (2.1) yields

$$\left(\sum_{n=1}^{\infty}(-1)^{n-1}\psi^n_\Lambda(V)t^{n-1}\right)\left(\sum_{n=0}^{\infty}\Lambda^n(V)t^n\right) = \sum_{n=1}^{\infty} n\Lambda^n(V)t^{n-1}.$$

By comparing coefficients of t^{n-1} we obtain Newton's formula

$$\sum_{j=0}^{n-1}(-1)^{j+1}\psi^{n-j}_\Lambda(V)\Lambda^j(V) \quad = \quad (-1)^n n\Lambda^n(V) \tag{2.4}$$

for all $n > 0$ and all KG-modules V. Hence

$$\psi^n_\Lambda(V) \quad = \quad \sum_{j=1}^{n-1}(-1)^{j+1}\psi^{n-j}_\Lambda(V)\Lambda^j(V) + (-1)^{n+1}n\Lambda^n(V). \tag{2.5}$$

A similar equation may be obtained for $\psi^n_S(V)$ by means of (2.2).

Lemma 2.2. *Let K be a field of prime characteristic p, G a finite p-group, V the regular KG-module and n a positive integer not divisible by p. Then*

$$\psi^n_\Lambda(V) = \psi^n_S(V) = V.$$

Proof. Since G is a p-group, V is indecomposable and $V^{\otimes n}$ is a direct multiple of V. Therefore, by Proposition 2.1 and (2.3), we have $\psi^n_\Lambda(V) = \psi^n_S(V) = V$. □

From now on we assume that G is a finite group. For any subgroup H of G and any KG-module V we write $V{\downarrow_H}$ to denote the KH-module obtained from V by restriction. Since restriction commutes with taking direct sums and tensor

products there is a ring homomorphism from R_{KG} to R_{KH} mapping V to $V{\downarrow_H}$ for every KG-module V. For each $A \in R_{KG}$ we write $A{\downarrow_H}$ for the image of A under this homomorphism. Since restriction commutes with the formation of exterior and symmetric powers, we have (see [4, Lemma 2.3])

$$\psi^n_\Lambda(A){\downarrow_H} = \psi^n_\Lambda(A{\downarrow_H}), \quad \psi^n_S(A){\downarrow_H} = \psi^n_S(A{\downarrow_H}), \tag{2.6}$$

for all $n > 0$ and all $A \in R_{KG}$. (Note that the Adams operations on the left-hand sides of these equations act on R_{KG} whilst those on the right act on R_{KH}.)

For any KH-module U we write $U{\uparrow^G}$ to denote the KG-module obtained from U by induction. Since induction commutes with taking direct sums there is a $\mathbb{Z}$-linear map from R_{KH} to R_{KG} that maps U to $U{\uparrow^G}$ for every KH-module U. For each $A \in R_{KH}$ we write $A{\uparrow^G}$ for the image of A under this map. For any KG-module V and any KH-module U we have $U{\uparrow^G} V = (U(V{\downarrow_H})){\uparrow^G}$ in R_{KG} (by [2, Proposition 3.3.3], for example). Hence the $\mathbb{Z}$-span of all modules induced from H is an ideal of R_{KG}. Similarly, the $\mathbb{Z}$-span of all relatively H-projective KG-modules is an ideal of R_{KG}. In particular, the $\mathbb{Z}$-span of all projective modules is an ideal of R_{KG}. For $A, B \in R_{KG}$ we write $A \overset{\text{proj}}{=} B$ to mean that $A - B$ is a $\mathbb{Z}$-linear combination of projectives.

For a KG-module V let $\mathcal{P}(V)$ denote a projective cover of V (see [2, Section 1.5] for further details). The Heller translate $\Omega(V)$ is then defined (up to isomorphism) as the kernel of the map $\mathcal{P}(V) \twoheadrightarrow V$. The definition of Ω extends to give a $\mathbb{Z}$-linear map $\Omega : R_{KG} \to R_{KG}$. Hence, for each $n \geqslant 0$, we have a $\mathbb{Z}$-linear map $\Omega^n : R_{KG} \to R_{KG}$ defined by composition: thus Ω^0 is the identity map and $\Omega^n = \Omega \circ \Omega^{n-1}$ for $n > 0$. It can be shown that $\Omega(AB) \overset{\text{proj}}{=} \Omega(A)B$ for all $A, B \in R_{KG}$ (see [2, Corollary 3.1.6]). Repeated application of this result yields

$$\Omega^i(A)\Omega^j(B) \overset{\text{proj}}{=} \Omega^{i+j}(AB) \tag{2.7}$$

for all $i, j \geqslant 0$ and all $A, B \in R_{KG}$.

For any subgroup H of G the map $\Omega : R_{KH} \to R_{KH}$ is defined in the same way as $\Omega : R_{KG} \to R_{KG}$ (but using KH-modules instead of KG-modules). It follows easily from Schanuel's lemma (see [2, Lemma 1.5.3]) that $\Omega(V){\downarrow_H} \overset{\text{proj}}{=} \Omega(V{\downarrow_H})$ for every KG-module V. Thus, for all $A \in R_{KG}$, we have

$$\Omega(A){\downarrow_H} \overset{\text{proj}}{=} \Omega(A{\downarrow_H}). \tag{2.8}$$

Let N be a normal subgroup of G. Each $K(G/N)$-module yields a KG-module by 'inflation'. Indeed, inflation yields a ring embedding $\mu : R_{K(G/N)} \to R_{KG}$. Also, since μ commutes with the formation of exterior and symmetric powers, we have

$$\psi^n_\Lambda(\mu(A)) = \mu(\psi^n_\Lambda(A)), \quad \psi^n_S(\mu(A)) = \mu(\psi^n_S(A)), \tag{2.9}$$

for all $n > 0$ and all $A \in R_{K(G/N)}$ (see [4, Lemma 2.3]).

3 Periodicity of the Adams operations ψ^n_Λ

Let G be a finite group and K a field. One of the main themes of this paper is the periodicity, as a function of n, of the Adams operations ψ^n_Λ and ψ^n_S on R_{KG}. When we refer to the periodicity of ψ^n_Λ or ψ^n_S we shall always mean periodicity in n.

Lemma 3.1. *Let G be a finite group and let K be a field such that $|G|$ is not divisible by the characteristic of K. Then $\psi^n_\Lambda = \psi^{n+e}_\Lambda$ and $\psi^n_S = \psi^{n+e}_S$, for all $n > 0$, where e is the exponent of G.*

Proof. (When $\operatorname{char} K = 0$ this result is essentially well known.) For any KG-module V let $\operatorname{Br}(V)$ denote the Brauer character of V. Thus $\operatorname{Br}(V) : G \to \mathbb{C}$. Hence we define $\operatorname{Br}(A) : G \to \mathbb{C}$ for all $A \in R_{KG}$ so that Br is $\mathbb{Z}$-linear on R_{KG}. Let V be a KG-module. Then, by [4, Lemma 2.6], we have

$$\operatorname{Br}(\psi^n_\Lambda(V))(g) = \operatorname{Br}(V)(g^n) = \operatorname{Br}(\psi^n_S(V))(g)$$

for all $g \in G$. Hence

$$\operatorname{Br}(\psi^n_\Lambda(V)) = \operatorname{Br}(\psi^{n+e}_\Lambda(V)) = \operatorname{Br}(\psi^{n+e}_S(V)) = \operatorname{Br}(\psi^n_S(V)).$$

However, for all $A, B \in R_{KG}$ we have $\operatorname{Br}(A) = \operatorname{Br}(B)$ if and only if $A = B$ in R_{KG} (as can be derived from [2, Corollary 5.3.6]). Hence the result follows. □

Under the assumptions of Lemma 3.1, e is, in fact, the minimum period of ψ^n_Λ and ψ^n_S, as we shall see in Section 7. When $\operatorname{char} K = 0$, Lemma 3.1 yields the periodicity of ψ^n_Λ and ψ^n_S for every finite group. Thus for the rest of this section we shall assume that K has prime characteristic p. Here we concentrate on ψ^n_Λ. The periodicity of ψ^n_S will be studied in Section 5.

It was proved in [4, Theorem 7.2] that ψ^n_Λ and ψ^n_S are periodic (with specified upper bounds for the periods) when the Sylow p-subgroups of G have order p.

Here we generalise this fact for ψ_Λ^n as follows. The proof is relatively simple but gives only a crude upper bound for the period.

Theorem 3.2. *Let G be a finite group and let K be a field of prime characteristic p. Then the Adams operations ψ_Λ^n on the Green ring R_{KG} are periodic in n if and only if the Sylow p-subgroups of G are cyclic.*

Proof. Suppose first that the Sylow p-subgroups of G are cyclic. By a theorem of Higman (see [6, Theorem (64.1)]), there are only finitely many isomorphism classes of indecomposable KG-modules. Hence $\mathbb{C}R_{KG}$ is finite-dimensional. By work of Green and O'Reilly (see [7, Theorem (81.90)]), $\mathbb{C}R_{KG}$ is semisimple. Thus $\mathbb{C}R_{KG}$ has a $\mathbb{C}$-basis $\{e_i : i = 1, \ldots, m\}$ consisting of pairwise-orthogonal idempotents. Note that every idempotent of $\mathbb{C}R_{KG}$ has the form $\sum_{j \in S} e_j$, where S is a subset of $\{1, \ldots, m\}$, and every endomorphism of $\mathbb{C}R_{KG}$ maps each e_i to an idempotent. Thus $\mathbb{C}R_{KG}$ has only finitely many endomorphisms. By Proposition 2.1, ψ_Λ^n is an endomorphism of $\mathbb{C}R_{KG}$ for all n not divisible by p. Thus there are only finitely many possibilities for the maps ψ_Λ^n where $p \nmid n$.

Choose a positive integer d such that $\dim V \leqslant p^d$ for every indecomposable KG-module V. If n is a positive integer not divisible by p^{d+1} we may write $n = p^i k$ where $0 \leqslant i \leqslant d$ and $p \nmid k$. Thus, by Proposition 2.1, $\psi_\Lambda^n = \psi_\Lambda^k \circ \psi_\Lambda^{p^i}$. There are only finitely many possibilities for ψ_Λ^k and at most $d+1$ possibilities for $\psi_\Lambda^{p^i}$. Hence there are only finitely many possibilities for the maps ψ_Λ^n where $p^{d+1} \nmid n$.

If $c = p^d k$ where $p \nmid k$ then $p^{d+1} \nmid n$ for all $n \in \{c, c+1, \ldots, c+p^d-1\}$. Hence there are only finitely many possibilities for the p^d-tuple $(\psi_\Lambda^c, \ldots, \psi_\Lambda^{c+p^d-1})$ where c has the given form. It follows that there exist positive integers a and b such that $a < b$ and

$$(\psi_\Lambda^a, \ldots, \psi_\Lambda^{a+p^d-1}) = (\psi_\Lambda^b, \ldots, \psi_\Lambda^{b+p^d-1}). \tag{3.1}$$

Write $s = b - a$. We shall show that $\psi_\Lambda^n = \psi_\Lambda^{n+s}$ for all $n > 0$.

Let V be any indecomposable KG-module and set $r = \dim V$. Thus $r \leqslant p^d$. It suffices to show that $\psi_\Lambda^n(V) = \psi_\Lambda^{n+s}(V)$ for all $n > 0$. For each n define an r-tuple Ψ_n by

$$\Psi_n = (\psi_\Lambda^n(V), \ldots, \psi_\Lambda^{n+r-1}(V)).$$

It suffices to show that $\Psi_n = \Psi_{n+s}$ for all $n > 0$. By (3.1), we have $\Psi_a = \Psi_{a+s}$. Thus it suffices to show, for all $n > 0$, that $\Psi_n = \Psi_{n+s}$ if and only if $\Psi_{n+1} = \Psi_{n+1+s}$.

Since $\Lambda^j(V) = 0$ for $j > r$, Newton's formulae, (2.4) and (2.5), with n replaced by $n + r$, become

$$\sum_{j=0}^{r}(-1)^{j+1}\psi_\Lambda^{n+r-j}(V)\Lambda^j(V) = 0 \tag{3.2}$$

and

$$\psi_\Lambda^{n+r}(V) = \sum_{j=1}^{r}(-1)^{j+1}\psi_\Lambda^{n+r-j}(V)\Lambda^j(V), \tag{3.3}$$

for all $n > 0$. Since $\Lambda^r(V)$ is one-dimensional, there exists a one-dimensional KG-module W such that $\Lambda^r(V)W = 1$ in R_{KG}. Hence, from (3.2), we obtain

$$(-1)^r\psi_\Lambda^n(V) = \sum_{j=0}^{r-1}(-1)^{j+1}\psi_\Lambda^{n+r-j}(V)\Lambda^j(V)W. \tag{3.4}$$

The elements $\psi_\Lambda^{n+r-j}(V)$ on the right of (3.3) are the components of Ψ_n (in reverse order). Hence if $\Psi_n = \Psi_{n+s}$ we obtain $\psi_\Lambda^{n+r}(V) = \psi_\Lambda^{n+r+s}(V)$ and hence $\Psi_{n+1} = \Psi_{n+1+s}$. Similarly the elements $\psi_\Lambda^{n+r-j}(V)$ on the right of (3.4) are the components of Ψ_{n+1}. Hence if $\Psi_{n+1} = \Psi_{n+1+s}$ we obtain $\psi_\Lambda^n(V) = \psi_\Lambda^{n+s}(V)$ and hence $\Psi_n = \Psi_{n+s}$. This gives the required result.

We shall now prove the converse, omitting some of the details for the sake of brevity. Let G be a finite group with non-cyclic Sylow p-subgroups. We prove that ψ_Λ^n is non-periodic. In fact we prove the stronger result that the ψ_Λ^n are distinct for $p \nmid n$. (Since $\psi_\Lambda^n = \psi_S^n$ when $p \nmid n$, our proof shows that ψ_S^n is also non-periodic.)

Let H be a minimal non-cyclic p-subgroup of G. Thus either $H \cong C_p \times C_p$ or $p = 2$ and $H \cong Q_8$, where C_p has order p and Q_8 is the quaternion group.

Suppose first that $H \cong C_p \times C_p$. Write K for the trivial one-dimensional KH-module, the identity element of R_{KH}. The Heller translates $\Omega^n(K)$, for $n \geqslant 1$, are isomorphic to the kernels of the maps in a minimal projective resolution of K. Since $H \cong C_p \times C_p$, this resolution is non-periodic. (If it is periodic its modules have bounded dimension and so the spaces $\mathrm{Ext}_{KH}^n(K, K)$ have bounded dimension. But $\mathrm{Ext}_{KH}^n(K, K) \cong H^n(H, K)$, and $H^n(H, K)$ has dimension $n + 1$ by the Künneth formula.) It follows that the $\Omega^n(K)$ are distinct elements of R_{KH} for $n \geqslant 1$. Furthermore, since K is a non-projective indecomposable, each $\Omega^n(K)$ is a non-projective indecomposable. Write $V = \Omega(K)$. Then, by (2.7), we have

$$\Omega^n(K) = \Omega^n(K^n) \overset{\text{proj}}{=} (\Omega(K))^n = V^n,$$

for all $n \geqslant 1$. Thus $V^{\otimes n}$ is isomorphic to a direct sum of $\Omega^n(K)$ and projectives.

Let n be a positive integer not divisible by p. Then, by Proposition 2.1, $\psi^n_\Lambda(V)$ is a $\mathbb{Z}$-linear combination of direct summands of $V^{\otimes n}$. However, $\dim V = p^2 - 1$ and so $\delta(\psi^n_\Lambda(V)) = p^2 - 1$ by (2.3). Thus $\psi^n_\Lambda(V)$ is not a $\mathbb{Z}$-linear combination of projectives only and $\psi^n_\Lambda(V)$ must involve $\Omega^n(K)$. Therefore, for values of n not divisible by p, the elements $\psi^n_\Lambda(V)$ of R_{KH} are distinct.

Let U be the Heller translate of the trivial one-dimensional KG-module. By (2.8) we have $U{\downarrow_H} \overset{\text{proj}}{=} V = \Omega(K)$. Since $\Omega(K)$ is a non-projective indecomposable, we find that $U{\downarrow_H} = V + W$, where W is projective.

Let m and n be distinct positive integers not divisible by p. Then, as seen above, $\psi^m_\Lambda(V) \neq \psi^n_\Lambda(V)$. However, by Lemma 2.2, $\psi^m_\Lambda(W) = \psi^n_\Lambda(W)$. Therefore $\psi^m_\Lambda(U{\downarrow_H}) \neq \psi^n_\Lambda(U{\downarrow_H})$. Hence $\psi^m_\Lambda(U) \neq \psi^n_\Lambda(U)$, by (2.6). Thus $\psi^m_\Lambda \neq \psi^n_\Lambda$ on R_{KG}.

Now suppose that $p = 2$ and $H \cong Q_8$. Thus H has a normal subgroup N such that $H/N \cong C_2 \times C_2$. Let V be the $K(H/N)$-module defined by $V = \Omega(K)$, as above. Thus V is three-dimensional and is isomorphic to the augmentation ideal of $K(H/N)$. As we have seen, the elements $\psi^n_\Lambda(V)$ of $R_{K(H/N)}$ are distinct for $p \nmid n$. We now regard V as a KH-module by inflation. Thus N acts trivially on V and, by (2.9), the elements $\psi^n_\Lambda(V)$ of R_{KH} are distinct for $p \nmid n$.

Let $M = V{\uparrow^G}$. By Mackey's decomposition theorem (see [2, Theorem 3.3.4]), $M{\downarrow_H}$ is isomorphic to a direct sum of modules $V(g)$, where g ranges over a set of representatives of double cosets HgH. Indeed, $V(g) = (V \otimes g){\downarrow_{H^g \cap H}}{\uparrow^H}$, where $V \otimes g$ is the $K(H^g)$-module 'conjugate' to the KH-module V in which we have $(v \otimes g)g^{-1}hg = vh \otimes g$ for all $v \in V$, $h \in H$.

Note that the subgroups of H are H, $\{1\}$, N, and cyclic subgroups L_1, L_2, L_3 of order 4, all of these subgroups being normal. Note also that H^g has only one element of order 2 and this acts trivially on $V \otimes g$. It is straightforward to check that if $H^g \cap H = H$ then $V(g) \cong V$. If $H^g \cap H = \{1\}$ then $V(g)$ is clearly a free KH-module. If $H^g \cap H = N$ then N acts trivially on $V \otimes g$ and so $V(g)$ is a free $K(H/N)$-module (regarded as a KH-module). If $H^g \cap H = L_i$ for $i \in \{1, 2, 3\}$ then it is straightforward to check that $(V \otimes g){\downarrow_{H^g \cap H}}$ is the direct sum of a one-dimensional trivial KL_i-module and a regular $K(L_i/N)$-module: hence $V(g)$ is the direct sum of a regular $K(H/L_i)$-module and a regular $K(H/N)$-module. It follows that $M{\downarrow_H} = rV + W$, where r is a positive integer and W is a sum of regular modules for factor groups of H regarded as KH-modules.

Let m and n be distinct positive integers not divisible by p. Then we have $\psi^m_\Lambda(rV) \neq \psi^n_\Lambda(rV)$. Also, by Lemma 2.2 and (2.9), we have $\psi^m_\Lambda(W) = \psi^n_\Lambda(W)$. It follows that $\psi^m_\Lambda(M) \neq \psi^n_\Lambda(M)$, and so $\psi^m_\Lambda \neq \psi^n_\Lambda$ on R_{KG}. $\square$

4 The Adams operations ψ^n_Λ for a cyclic p-group

In this section we shall consider the Adams operations ψ^n_Λ on the Green ring R_{KC}, where K is a field of prime characteristic p and C is a cyclic p-group of order $q \geqslant 1$. As we have seen in Theorem 3.2, ψ^n_Λ is periodic in n. Here we shall show, for $q > 1$, that the minimum period is $2q$. We shall also establish the symmetry property that $\psi^n_\Lambda = \psi^{2q-n}_\Lambda$ for all $n < 2q$.

There are precisely q indecomposable KC-modules up to isomorphism, which we denote by $V_1, \ldots, V_q$, where V_r has dimension r for $r = 1, \ldots, q$. Thus R_{KC} has $\mathbb{Z}$-basis $\{V_1, \ldots, V_q\}$. The one-dimensional module V_1 is the identity element of R_{KC}, which we shall sometimes write simply as 1. When $q = 1$ we have $R_{KC} = \mathbb{Z}V_1$ so that ψ^n_Λ is the identity map for all $n > 0$. Thus we usually assume that $q > 1$.

For $q > 1$ let $\tilde{C}$ denote the subgroup of index p in C. Thus $\tilde{C}$ is cyclic of order q/p. We write $\tilde{V}_1, \ldots, \tilde{V}_{q/p}$ to denote the indecomposable $K\tilde{C}$-modules, up to isomorphism, where $\tilde{V}_r$ has dimension r, for $r = 1, \ldots, q/p$. Let $r \in \{1, \ldots, q\}$. Then it is well known (and easy to prove) that

$$V_r{\downarrow_{\tilde{C}}} = (p-b)\tilde{V}_a + b\tilde{V}_{a+1}, \tag{4.1}$$

where $r = ap + b$ with $0 \leqslant b < p$. (We take the convention that $\tilde{V}_0 = 0$.) Notice that $a + 1 \leqslant q/p$, provided that $b > 0$. Also, for $r = 1, \ldots, q/p$, we have

$$\tilde{V}_r{\uparrow^C} = V_{pr}. \tag{4.2}$$

A KC-module will be said to be induced if it is induced from a $K\tilde{C}$-module, and an element of R_{KC} will be said to be induced if it is a $\mathbb{Z}$-linear combination of induced modules. The set of induced elements is an ideal of R_{KC} and, by (4.2), this ideal has $\mathbb{Z}$-basis $\{V_p, V_{2p}, \ldots, V_q\}$. For $A, B \in R_{KC}$ we write $A \overset{\text{ind}}{=} B$ to mean that $A - B$ is induced.

Lemma 4.1. *Let $q > 1$ and $A \in R_{KC}$.*
(i) *If $A \overset{\text{ind}}{=} 0$ and $A{\downarrow_{\tilde{C}}} = 0$ then $A = 0$.*
(ii) *If $A \overset{\text{ind}}{=} 0$ and $A{\downarrow_{\tilde{C}}} \overset{\text{proj}}{=} 0$ then $A \overset{\text{proj}}{=} 0$.*

Proof. (i) Suppose that $A \overset{\text{ind}}{=} 0$ and $A\downarrow_{\tilde{C}} = 0$. Since $A \overset{\text{ind}}{=} 0$ we may write $A = \sum_{r=1}^{q/p} \alpha_r V_{pr}$, where $\alpha_r \in \mathbb{Z}$ for $r = 1, \dots, q/p$. Since $A\downarrow_{\tilde{C}} = 0$ we have $\sum_{r=1}^{q/p} \alpha_r p\tilde{V}_r = 0$ by (4.1). Thus $\alpha_r = 0$ for all r, and hence $A = 0$.

(ii) This is similar. □

The regular module V_q is the unique projective indecomposable KC-module. Thus, for $A, B \in R_{KC}$, we have $A \overset{\text{proj}}{=} B$ if and only if $A - B \in \mathbb{Z}V_q$. For $q > 1$ we have that V_q is induced and so $A \overset{\text{proj}}{=} B$ implies $A \overset{\text{ind}}{=} B$ for all $A, B \in R_{KC}$.

If $1 \leqslant p^j \leqslant q$ then C has a factor group $C(p^j)$ of order p^j and inflation of modules gives a ring embedding $R_{KC(p^j)} \to R_{KC}$. Thus, for $r = 1, \dots, p^j$, we may identify V_r with the indecomposable $KC(p^j)$-module of dimension r. Since V_{p^j} is the only projective indecomposable $KC(p^j)$-module we have

$$V_r V_{p^j} = rV_{p^j}, \quad \text{for } r = 1, \dots, p^j. \tag{4.3}$$

Let P_{KC} denote the $\mathbb{Z}$-span in R_{KC} of the set of all permutation modules for C over K. Each transitive permutation module M is the module induced from the one-dimensional trivial module for some subgroup of C (namely, the stabilizer of a point). Thus, by repeated use of (4.2) (applied to subgroups of C rather than C itself), we find that $M \cong V_{p^j}$ for some j such that $1 \leqslant p^j \leqslant q$. Conversely, the modules V_{p^j} are permutation modules. Thus P_{KC} has $\mathbb{Z}$-basis $\{V_1, V_p, V_{p^2}, \dots, V_q\}$. Since the tensor product of permutation modules is a permutation module, P_{KC} is a subring of R_{KC}. (This also follows from (4.3).)

For $A \in R_{KC}$ write $A = \sum_{i=1}^{q} \alpha_i(A)V_i$, where $\alpha_i(A) \in \mathbb{Z}$ for $i = 1, \dots, q$. Thus $\alpha_i(A)$ denotes the multiplicity with which V_i occurs in A. The following result is an easy consequence of Lemma 4.1 (i).

Corollary 4.2. *Let $q > 1$ and let $A, B \in P_{KC}$. If $\alpha_1(A) = \alpha_1(B)$ and $A\downarrow_{\tilde{C}} = B\downarrow_{\tilde{C}}$ then $A = B$.*

Let $r \in \{1, \dots, q\}$ and $j \in \{0, \dots, r\}$. Then $\dim \Lambda^j(V_r) = \dim \Lambda^{r-j}(V_r)$. In fact

$$\Lambda^j(V_r) = \Lambda^{r-j}(V_r) \tag{4.4}$$

in R_{KC}. This is well known but we sketch a proof. Since $\dim \Lambda^r(V_r) = 1$ we have $\Lambda^r(V_r) \cong K$ as a KC-module. Also there is an isomorphism of K-spaces, $\theta : \Lambda^j(V_r) \to \mathrm{Hom}_K(\Lambda^{r-j}(V_r), \Lambda^r(V_r))$, induced by multiplication in the exterior

algebra $\Lambda(V_r)$. We may regard $\mathrm{Hom}_K(\Lambda^{r-j}(V_r), \Lambda^r(V_r))$ as a KC-module, the contragredient dual of $\Lambda^{r-j}(V_r)$. Then it is easily verified that θ is an isomorphism of KC-modules. Hence (4.4) follows because all KC-modules are self-dual.

Lemma 4.3. *For each non-negative integer n we have $\Lambda^n(V_q) \in P_{KC}$ and*

$$\alpha_1(\Lambda^n(V_q)) = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = q, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $\Lambda^0(V_q) = \Lambda^q(V_q) = V_1$ and $\Lambda^n(V_q) = 0$ for $n > q$, the result holds for $n = 0$ and for all $n \geqslant q$. Thus we assume that $1 \leqslant n < q$.

Let g be a generator of C. Then the regular module V_q has basis $\{x_1, \ldots, x_q\}$ where $x_i g = x_{i+1}$ for $1 \leqslant i < q$ and $x_q g = x_1$. It is easily checked that the set

$$\{(-1)^{(i_1+\cdots+i_n)(n+1)} x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n} : 1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant q\}$$

is a basis of $\Lambda^n(V_q)$ invariant under the action of g. Thus $\Lambda^n(V_q)$ is a permutation module, and so $\Lambda^n(V_q) \in P_{KC}$. Since $n < q$ there are no orbits of length one in the given basis. Hence $\alpha_1(\Lambda^n(V_q)) = 0$. □

Lemma 4.4. *Let $q > 1$ and $n > 0$.*
(i) *We have $\psi^n_\Lambda(V_q) \in P_{KC}$.*
(ii) *If p is odd then*

$$\alpha_1(\psi^n_\Lambda(V_q)) = \begin{cases} q & \text{if } q \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *If $p = 2$ then*

$$\alpha_1(\psi^n_\Lambda(V_q)) = \begin{cases} (-1)^{n/q} q & \text{if } q \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We prove the result by induction on n. The result is trivial for $n = 1$ because $\psi^1_\Lambda(V_q) = V_q$. Now let $n > 1$ and suppose that the result holds for all k such that $1 \leqslant k < n$. By (2.5) we have that

$$\psi^n_\Lambda(V_q) = \sum_{j=1}^{n-1} (-1)^{j+1} \psi^{n-j}_\Lambda(V_q) \Lambda^j(V_q) + (-1)^{n+1} n \Lambda^n(V_q). \tag{4.5}$$

By our inductive hypothesis, $\psi^{n-j}_\Lambda(V_q) \in P_{KC}$ for $j = 1, \ldots, n-1$. Also, by Lemma 4.3, $\Lambda^j(V_q) \in P_{KC}$ for $j = 1, \ldots, n$. Since P_{KC} is a subring of R_{KC} it now follows from (4.5) that $\psi^n_\Lambda(V_q) \in P_{KC}$, as required for (i).

For all $k > 0$ let $\beta_k = \alpha_1(\Lambda^k(V_q))$ and $\gamma_k = \alpha_1(\psi^k_\Lambda(V_q))$. Then, by (4.5) and (4.3), we have

$$\gamma_n = \sum_{j=1}^{n-1}(-1)^{j+1}\gamma_{n-j}\beta_j + (-1)^{n+1}n\beta_n.$$

By Lemma 4.3 it follows that $\gamma_n = 0$ if $n < q$, $\gamma_q = (-1)^{q+1}q$ and $\gamma_n = (-1)^{q+1}\gamma_{n-q}$ if $n > q$. Parts (ii) and (iii) now follow from our inductive hypothesis. □

For all positive integers a and b, let (a, b) denote the greatest common divisor of a and b.

Proposition 4.5. *Let $n > 0$.*
(i) *If p is odd then $\psi^n_\Lambda(V_q) = (n, q)V_{q/(n,q)}$.*
(ii) *If $p = 2$ then*

$$\psi^n_\Lambda(V_q) = \begin{cases} V_q & \text{if } n \text{ is odd,} \\ (n, 2q)V_{2q/(n,2q)} - (n, q)V_{q/(n,q)} & \text{if } n \text{ is even.} \end{cases}$$

Proof. When $q = 1$, (i) and (ii) hold trivially. From now on we assume that $q > 1$ and write $\tilde{q} = q/p$. Define $A_n = (n, q)V_{q/(n,q)}$ and $\tilde{A}_n = (n, \tilde{q})\tilde{V}_{\tilde{q}/(n,\tilde{q})}$. By considering the cases where $q \mid n$ and $q \nmid n$ separately, it is easily verified that

$$A_n{\downarrow_{\tilde{C}}} = p\tilde{A}_n. \tag{4.6}$$

It is also easy to see that

$$\alpha_1(A_n) = \begin{cases} q & \text{if } q \mid n, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

(i) Suppose that p is odd. It suffices to show that $\psi^n_\Lambda(V_q) = A_n$. By Lemma 4.4 (ii) and (4.7) we have $\alpha_1(\psi^n_\Lambda(V_q)) = \alpha_1(A_n)$. By (2.6) and (4.1) we have $\psi^n_\Lambda(V_q){\downarrow_{\tilde{C}}} = p\psi^n_\Lambda(\tilde{V}_{\tilde{q}})$. Also we may assume by induction on q that $\psi^n_\Lambda(\tilde{V}_{\tilde{q}}) = \tilde{A}_n$. Thus, by (4.6), we have $\psi^n_\Lambda(V_q){\downarrow_{\tilde{C}}} = A_n{\downarrow_{\tilde{C}}}$. However, $\psi^n_\Lambda(V_q) \in P_{KC}$, by Lemma 4.4 (i), and, clearly, $A_n \in P_{KC}$. The result now follows by Corollary 4.2.

(ii) Suppose that $p = 2$. Define $B_n = V_q$ if n is odd and $B_n = 2A_{n/2} - A_n$ if n is even, and define $\tilde{B}_n$ similarly. Then $B_n{\downarrow_{\tilde{C}}} = 2\tilde{B}_n$, by (4.6), and it suffices to show that $\psi^n_\Lambda(V_q) = B_n$. Separating the cases $(n, 2q) = 2q$, $(n, 2q) = q$ and $(n, 2q) < q$ in Lemma 4.4 (iii) and (4.7), we obtain $\alpha_1(\psi^n_\Lambda(V_q)) = \alpha_1(B_n)$. By (2.6)

and (4.1) we have $\psi^n_\Lambda(V_q){\downarrow_{\tilde{C}}} = 2\psi^n_\Lambda(\tilde{V}_{\tilde{q}})$. Also we may assume by induction that $\psi^n_\Lambda(\tilde{V}_{\tilde{q}}) = \tilde{B}_n$. Thus $\psi^n_\Lambda(V_q){\downarrow_{\tilde{C}}} = B_n{\downarrow_{\tilde{C}}}$. The result follows by Corollary 4.2. $\square$

In Sections 6 and 7 we shall also need the corresponding result for $\psi^n_S(V_q)$.

Proposition 4.6. *For all $n > 0$ we have $\psi^n_S(V_q) = (n, q)V_{q/(n,q)}$.*

Proof. It is easy to verify that $S^n(V_q)$ is a permutation module with respect to the usual basis. Counting orbits of length one yields that $\alpha_1(S^n(V_q)) = 1$ if $q \mid n$ and $\alpha_1(S_n(V_q)) = 0$ otherwise. By arguments similar to the proofs of Lemma 4.4 (i) and (ii), we deduce that $\psi^n_S(V_q) \in P_{KC}$ where $\alpha_1(\psi^n_S(V_q)) = q$ if $q \mid n$ and $\alpha_1(\psi^n_S(V_q)) = 0$ otherwise. The result then follows by an argument similar to the proof of Proposition 4.5. $\square$

Proposition 4.7. *For all $n > 0$ such that $q \nmid n$ we have $\psi^n_\Lambda = \psi^{n+2p(n,q)}_\Lambda$.*

Proof. If $(n, q) = 1$ then, by [5, Theorem 3.3], we have $\psi^n_\Lambda = \psi^{n+2p}_\Lambda$, as required. So suppose that $1 < (n, q) < q$. Then $n = (n, q)k$, where $p \nmid k$, and hence $n + 2p(n, q) = (n, q)(k + 2p)$, where $p \nmid k + 2p$. Thus $\psi^n_\Lambda = \psi^k_\Lambda \circ \psi^{(n,q)}_\Lambda$ and $\psi^{n+2p(n,q)}_\Lambda = \psi^{k+2p}_\Lambda \circ \psi^{(n,q)}_\Lambda$, by Proposition 2.1. Since $p \nmid k$ we have $\psi^k_\Lambda = \psi^{k+2p}_\Lambda$. Thus $\psi^n_\Lambda = \psi^{n+2p(n,q)}_\Lambda$. $\square$

Recall that δ is the endomorphism of R_{KC} satisfying $\delta(V_r) = rV_1$ for all r.

Theorem 4.8. *Let K be a field of prime characteristic p and let C be a cyclic p-group of order $q > 1$.*
(i) *The Adams operations ψ^n_Λ on R_{KC} are periodic in n, with minimum period $2q$.*
(ii) *We have $\psi^n_\Lambda = \psi^{2q-n}_\Lambda$ for all $n < 2q$.*
(iii) *We have $\psi^{2q}_\Lambda = \delta$.*

Proof. (i) We begin by proving, by induction on n, that $\psi^n_\Lambda = \psi^{n+2q}_\Lambda$ for all $n > 0$. By Proposition 4.7, the result holds for all n such that $q \nmid n$. In particular the result holds for all $n < q$. Now assume that $n \geqslant q$ and that $\psi^k_\Lambda = \psi^{k+2q}_\Lambda$ for all $k < n$. Let $r \in \{1, \ldots, q\}$. It is enough to prove that $\psi^n_\Lambda(V_r) = \psi^{n+2q}_\Lambda(V_r)$.

For $r = q$ this follows easily from Proposition 4.5. Thus we may take $r < q$. Since $n \geqslant q > r$ and $\Lambda^j(V_r) = 0$ for $j > r$, (2.5) gives

$$\psi^n_\Lambda(V_r) = \sum_{j=1}^{r}(-1)^{j+1}\psi^{n-j}_\Lambda(V_r)\Lambda^j(V_r)$$

and

$$\psi_\Lambda^{n+2q}(V_r) = \sum_{j=1}^{r} (-1)^{j+1} \psi_\Lambda^{n+2q-j}(V_r)\Lambda^j(V_r).$$

Thus, by our inductive hypothesis, $\psi_\Lambda^n(V_r) = \psi_\Lambda^{n+2q}(V_r)$.

We have shown that ψ_Λ^n is periodic in n, with minimum period λ dividing $2q$. It remains to prove that $\lambda = 2q$. First suppose that $p = 2$. By Proposition 4.5 (ii), we see that $\psi_\Lambda^q(V_q) = \psi_\Lambda^{q+s}(V_q)$ if and only if s is a multiple of $2q$. Hence $2q$ divides λ, giving $\lambda = 2q$. Now suppose that p is odd. By Proposition 4.5 (i), we see that $\psi_\Lambda^q(V_q) = \psi_\Lambda^{q+s}(V_q)$ if and only if s is a multiple of q. Hence q divides λ, giving that $\lambda = q$ or $\lambda = 2q$. Since p is odd, $q + 1$ is even. Thus, by [5, Theorem 5.1], we find that $\psi_\Lambda^1(V_2) \neq \psi_\Lambda^{q+1}(V_2)$. Hence $\lambda = 2q$.

(ii) The result is trivial for $n = q$. By symmetry, it remains only to prove the result for $n < q$. Then we may write $n = (n, q)k$ where $(n, q) < q$ and $p \nmid k$. Also, $2q - n = (n, q)k'$, where $k' = 2q/(n, q) - k$. Since p divides $2q/(n, q)$ we have $p \nmid k'$. Thus by Proposition 2.1 we have $\psi_\Lambda^n = \psi_\Lambda^k \circ \psi_\Lambda^{(n,q)}$ and $\psi_\Lambda^{2q-n} = \psi_\Lambda^{k'} \circ \psi_\Lambda^{(n,q)}$. Since $k' \equiv -k \pmod{2p}$, we have $\psi_\Lambda^k = \psi_\Lambda^{k'}$ by [5, Corollary 3.4]. Thus $\psi_\Lambda^n = \psi_\Lambda^{2q-n}$.

(iii) It suffices to prove that $\psi_\Lambda^{2q}(V_r) = rV_1$ for $r \in \{1, \dots, q\}$. By (2.5),

$$\psi_\Lambda^{2q}(V_r) = \sum_{j=1}^{r} (-1)^{j+1} \psi_\Lambda^{2q-j}(V_r)\Lambda^j(V_r),$$

and, by (ii), we have $\psi_\Lambda^{2q-j} = \psi_\Lambda^j$ for all j such that $1 \leqslant j < 2q$. Moreover, by (4.4), we have $\Lambda^j(V_r) = \Lambda^{r-j}(V_r)$ for all j such that $1 \leqslant j \leqslant r$. Hence

$$\psi_\Lambda^{2q}(V_r) = \sum_{j=1}^{r} (-1)^{j+1} \psi_\Lambda^j(V_r)\Lambda^{r-j}(V_r) = (-1)^r \sum_{k=0}^{r-1} (-1)^{k+1} \psi_\Lambda^{r-k}(V_r)\Lambda^k(V_r).$$

Thus, by (2.4), we see that $\psi_\Lambda^{2q}(V_r) = r\Lambda^r(V_r) = rV_1$, as required. □

5 Periodicity of the Adams operations ψ_S^n

In this section we investigate the periodicity of ψ_S^n for the Green ring of an arbitrary finite group G over a field K. By Lemma 3.1, ψ_S^n is periodic if the characteristic of K does not divide $|G|$. Thus we assume that K has prime characteristic p. We shall show that ψ_S^n is periodic if and only if G has cyclic Sylow p-subgroups.

The main step in the proof is to establish that ψ_S^n is periodic in the special case when G has a normal cyclic Sylow p-subgroup with cyclic factor group. We

assume that G has this form for almost all of this section: only in Theorem 5.6 will we go to the general situation, using Conlon's induction theorem. We now fix some hypotheses and notation that will apply until Theorem 5.6.

Assume that K is algebraically closed. Let G be a finite group with a normal cyclic Sylow p-subgroup C such that G/C is cyclic. Thus (by the Schur-Zassenhaus theorem) G has a cyclic p'-subgroup H such that $G = HC$. Write $q = |C|$ and $m = |G/C| = |H|$, and let $C = \langle g \rangle$ and $H = \langle h \rangle$. By Theorem 3.2 there exists a positive integer π such that $\psi_\Lambda^n = \psi_\Lambda^{n+\pi}$ holds on R_{KF} for all $n > 0$ and for every subgroup F of G. However, when $p \nmid n$, we have $\psi_\Lambda^n = \psi_S^n$. Thus we have the following result.

Lemma 5.1. *There exists a positive integer π satisfying*
(i) *π is divisible by p and m,*
(ii) *$\psi_S^n = \psi_S^{n+\pi}$ on R_{KF} for all n such that $p \nmid n$ and for every subgroup F of G.*

All KH-modules will be regarded as KG-modules via the epimorphism $G \twoheadrightarrow H$ that restricts to the identity on H. Thus R_{KH} becomes a subring of R_{KG}. We consider the indecomposable KG-modules, as described in [1, pp. 34–37, 42–43]. We summarise the main facts, but adapt the presentation in [1] and use different notation. (In particular we use right modules instead of left modules.)

First note that, since H is cyclic and K is algebraically closed, there is a faithful one-dimensional KH-module X and the irreducible KH-modules up to isomorphism are the tensor powers $1, X, \ldots, X^{m-1}$. (We shall usually use Green ring notation for modules.) Clearly X has multiplicative order m in R_{KH}. The irreducible KG-modules are the same as the irreducible KH-modules.

When $q > 1$ let κ be an integer such that $h^{-1}gh = g^\kappa$ and reduce κ modulo p to obtain a non-zero element $\bar{\kappa}$ of the base field of K. Let W be the one-dimensional KH-module on which h acts as the scalar $\bar{\kappa}$. (This coincides with W as used in [1] and as described in [1, Exercise 5.3].) Also, let d be the multiplicative order of $\bar{\kappa}$. Thus $d \mid m$ and $d \mid p-1$. By suitable choice of X we can take $W = X^{m/d}$. Note that W has multiplicative order d in R_{KH}. When $q = 1$ we take $d = 1$ and $W = 1$.

Let J_q be the projective cover of the one-dimensional trivial KG-module. Then (see [1, p. 37]) J_q is uniserial with proper submodules $J_q(g-1)^r$, for $r = 1, \ldots, q$, and composition factors $1, W, W^2, \ldots, W^{q-1}$, from top to bottom. For $r = 1, \ldots, q$, write $J_r = J_q/J_q(g-1)^r$. It is easily verified that $J_r{\downarrow_C} = V_r$ in the notation of

Section 4. For $i = 0, \dots, m-1$, the (tensor product) module $X^i J_r$ is uniserial, and its composition factors from top to bottom are $X^i, X^i W, \dots, X^i W^{r-1}$. The modules $X^i J_r$, for $i = 0, \dots, m-1$ and $r = 1, \dots, q$, are the indecomposable KG-modules up to isomorphism (see [1, p. 42]). Note also that $(X^i J_r){\downarrow_C} = V_r$.

When $q > 1$ let $\tilde{C}$ denote the subgroup of index p in C, that is, $\tilde{C} = \langle g^p \rangle$. For $s = 1, \dots, q/p$, let $\tilde{V}_s$ denote the indecomposable $K\tilde{C}$-module of dimension s, as in Section 4.

Lemma 5.2. *When $q > 1$ the indecomposable KG-modules that are projective relative to $\tilde{C}$ are those of the form $X^i J_{ps}$ with $0 \leqslant i \leqslant m-1$ and $1 \leqslant s \leqslant q/p$.*

Proof. The module $X^i J_{ps}$ is relatively C-projective (by [1, Theorem 9.2], for example). Hence $X^i J_{ps}$ is a summand of $V_r{\uparrow^G}$ for some r. By Mackey's decomposition theorem, $V_r{\uparrow^G}{\downarrow_C} = mV_r$, and so we must have that $r = ps$. However, $V_{ps} = \tilde{V}_s{\uparrow^C}$. Therefore $X^i J_{ps}$ is a summand of $\tilde{V}_s{\uparrow^G}$ and is relatively $\tilde{C}$-projective.

Conversely, suppose that $X^i J_r$ is relatively $\tilde{C}$-projective. Then $X^i J_r$ is a summand of $\tilde{V}_s{\uparrow^G}$ for some s. Hence $X^i J_r$ is a summand of $V_{ps}{\uparrow^G}$. Thus, by Mackey's decomposition theorem, $r = ps$. $\square$

When $q > 1$ let $\tilde{G} = H\tilde{C} < G$ and let $\tilde{J}_1, \dots, \tilde{J}_{q/p}$ be the $K\tilde{G}$-modules defined in the same way as we defined $J_1, \dots, J_q$ for KG. Thus the indecomposable $K\tilde{G}$-modules are the modules $X^i \tilde{J}_s$ for $i = 0, \dots, m-1$ and $s = 1, \dots, q/p$.

Lemma 5.3. *When $q > 1$ we have that $J_{ps}{\downarrow_{\tilde{G}}} = (1 + W + \dots + W^{p-1})\tilde{J}_s$ for $s = 1, \dots, q/p$.*

Proof. Write $M = J_{ps}{\downarrow_{\tilde{G}}}$ and let $X^i \tilde{J}_k$ be any indecomposable summand of M. Since $M{\downarrow_{\tilde{C}}} = p\tilde{V}_s$, we must have $k = s$. Thus we may write $M = U\tilde{J}_s$, where U is a KH-module. We have $M/\mathrm{rad}(M) = U$ in $R_{K\tilde{G}}$, because $\tilde{J}_s/\mathrm{rad}(\tilde{J}_s)$ is trivial. However, $\mathrm{rad}(M) = M(g^p - 1)$, by [1, Lemma 5.8] applied to $\tilde{G}$. Since $J_{ps}(g^p - 1) = J_{ps}(g-1)^p$, the composition factors of $J_{ps}/J_{ps}(g^p - 1)$ are $1, W, \dots, W^{p-1}$. However $(J_{ps}/J_{ps}(g^p - 1)){\downarrow_{\tilde{G}}} = M/\mathrm{rad}(M)$ and $M/\mathrm{rad}(M)$ is completely reducible. Thus $U = 1 + W + \dots + W^{p-1}$. Hence $M = (1 + W + \dots + W^{p-1})\tilde{J}_s$, as required. $\square$

Assume that $q > 1$. In the Green ring R_{KG} the set I of all $\mathbb{Z}$-linear combinations of relatively $\tilde{C}$-projective KG-modules is an ideal (see Section 2). For

$A, B \in R_{KG}$ we write $A \approx B$ to mean that $A - B \in I$. Note that when $G = C$ we have $A \approx B$ if and only if $A \stackrel{\text{ind}}{=} B$. Thus the following result is a generalisation of Lemma 4.1 (i).

Lemma 5.4. *Let $q > 1$ and let A be an element of R_{KG} such that $A \approx 0$ and $A{\downarrow_{\tilde{G}}} = 0$. Then $A = 0$.*

Proof. By Lemma 5.2 we may write $A = \sum_s U_s J_{ps}$ where s ranges over $\{1, \ldots, q/p\}$ and $U_s \in R_{KH}$ for all s. By Lemma 5.3 and the assumption $A{\downarrow_{\tilde{G}}} = 0$ we obtain

$$\sum_s U_s(1 + W + \cdots + W^{p-1})\tilde{J}_s = 0$$

in $R_{K\tilde{G}}$. By the remark before Lemma 5.3 it follows that we have

$$U_s(1 + W + \cdots + W^{p-1}) = 0 \tag{5.1}$$

in R_{KH} for all $s \in \{1, \ldots, q/p\}$. It suffices to prove that $U_s = 0$. Recall that W has multiplicative order d, where $d \mid p - 1$. Thus, by (5.1), we have

$$\begin{aligned} U_s &= -U_s(1 + W + \cdots + W^{p-2}) \\ &= -((p-1)/d)U_s(1 + W + \cdots + W^{d-1}). \end{aligned} \tag{5.2}$$

Recall that $\{1, X, \ldots, X^{m-1}\}$ is a basis of R_{KH}. Since $W = X^{m/d}$ this basis may be written as $\{X^i W^j : 0 \leqslant i \leqslant l - 1,\ 0 \leqslant j \leqslant d - 1\}$, where $l = m/d$. Hence there are integers $n_{i,j}$ such that

$$-((p-1)/d)U_s = \sum_{i=0}^{l-1} \sum_{j=0}^{d-1} n_{i,j} X^i W^j.$$

Since $W^j(1 + W + \cdots + W^{d-1}) = (1 + W + \cdots + W^{d-1})$ for all j, we see, by (5.2), that there exist integers $n_0, n_1, \ldots, n_{l-1}$ such that

$$U_s = \sum_{i=0}^{l-1} n_i X^i (1 + W + \cdots + W^{d-1}). \tag{5.3}$$

Let $\phi : R_{KH} \to R_{KH}$ be the ring endomorphism defined by $\phi(X) = X^d$. Then $\phi(W) = X^m = 1$. Applying ϕ to (5.1) yields $p\phi(U_s) = 0$ and hence $\phi(U_s) = 0$. Thus, by applying ϕ to (5.3), we obtain the equation $0 = d\sum_{i=0}^{l-1} n_i X^{di}$ in R_{KH}. Therefore $n_i = 0$ for all i, and so $U_s = 0$, as required. $\square$

Suppose that $q > 1$ and recall that for $A, B \in R_{KG}$ we write $A \approx B$ to mean that $A - B \in I$. Similarly, for $A, B \in \mathbb{Q}R_{KG}$ we write $A \approx B$ to mean that $A - B \in \mathbb{Q}I$. For $A(t), B(t) \in \mathbb{Q}R_{KG}[[t]]$, where $A(t) = \sum_{i=0}^{\infty} A_i t^i$ and $B(t) = \sum_{i=0}^{\infty} B_i t^i$, we write $A(t) \approx B(t)$ to mean that $A_i \approx B_i$ for all i. It is easy to check that if $A_0 = B_0 = 1$ then $A(t) \approx B(t)$ implies $\log A(t) \approx \log B(t)$.

Let V be an indecomposable KG-module. Thus $V = X^i J_r$ for some i and r. By [8, Theorem 1.2], there is a one-dimensional submodule E of $S^q(V)$ such that

$$S(V,t) \approx (1 + B_1 t + \cdots + B_{q-1} t^{q-1})(1 + Et^q + E^2 t^{2q} + \cdots),$$

where $B_n = S^n(V)$ for $n = 1, \ldots, q-1$. Thus, by the remark above, we have

$$\log S(V,t) \approx \log((1 + B_1 t + \cdots + B_{q-1} t^{q-1})(1 + Et^q + E^2 t^{2q} + \cdots)). \tag{5.4}$$

Write $\psi_S(V,t) = \sum_{n=1}^{\infty} \psi_S^n(V) t^n$. Then, by (2.2), we have

$$\begin{aligned}
\psi_S(V,t) &= t\frac{d}{dt}\log S(V,t) \\
&\approx t\frac{d}{dt}\log(1 + B_1 t + \cdots + B_{q-1}t^{q-1}) - t\frac{d}{dt}\log(1 - Et^q) \qquad (5.5)\\
&\approx \frac{B_1 t + 2B_2 t^2 + \cdots + (q-1)B_{q-1}t^{q-1}}{1 + B_1 t + \cdots + B_{q-1}t^{q-1}} + \frac{qEt^q}{1 - Et^q}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
(\psi_S(V,t) - qEt^q(1 - Et^q)^{-1})&(1 + B_1 t + \cdots + B_{q-1}t^{q-1}) \\
&\approx B_1 t + 2B_2 t^2 + \cdots + (q-1)B_{q-1}t^{q-1},
\end{aligned}$$

so that

$$(D_1 t + D_2 t^2 + \cdots)(1 + B_1 t + \cdots + B_{q-1}t^{q-1}) \approx B_1 t + 2B_2 t^2 + \cdots + (q-1)B_{q-1}t^{q-1},$$

where

$$D_n = \begin{cases} \psi_S^n(V) & \text{if } q \nmid n, \\ \psi_S^n(V) - qE^{n/q} & \text{if } q \mid n. \end{cases} \tag{5.6}$$

Thus, for all $n \geqslant q$, we have

$$D_n + D_{n-1}B_1 + \cdots + D_{n-q+1}B_{q-1} \approx 0. \tag{5.7}$$

Proposition 5.5. *Let K be an algebraically closed field of prime characteristic p and let G be a finite group with a normal cyclic Sylow p-subgroup C such that G/C is cyclic, with $|C| = q$ and $|G/C| = m$. Let π be any positive integer satisfying the conditions of Lemma* 5.1*. Then $\psi^n_S = \psi^{n+q\pi/p}_S$ on R_{KG} for all $n > 0$.*

Proof. First note that $mp \mid \pi$ by Lemma 5.1 (i). Thus if $q = 1$ the result follows by Lemma 3.1. Now suppose that $q > 1$. We use the notation introduced above and we may assume, by induction, that the result holds for $\tilde{G} = H\tilde{C} < G$. Let V be an indecomposable KG-module and let $n > 0$. It suffices to show that $\psi^n_S(V) = \psi^{n'}_S(V)$ where $n' = n + q\pi/p$.

By the inductive hypothesis, $(\psi^n_S(V) - \psi^{n'}_S(V))\downarrow_{\tilde{G}} = 0$. We shall show that $\psi^n_S(V) \approx \psi^{n'}_S(V)$. This gives $\psi^n_S(V) - \psi^{n'}_S(V) = 0$ by Lemma 5.4, as required.

We use the notation of (5.4) and (5.6). Since E is a one-dimensional KG-module, E is a KH-module and $E^m = 1$ in R_{KG}. Since π/p is divisible by m we have $E^{n/q} = E^{(n+q\pi/p)/q}$ when $q \mid n$. Hence, by (5.6), the proposition will follow if we can prove that $D_n \approx D_{n'}$.

First suppose that $n < q$ and write $n = p^i k$ where $p \nmid k$. Thus $p^{i+1} \mid q$ and we have $n' = p^i k'$, where $k' = k + q\pi/p^{i+1}$. Since $p \mid \pi$ we have $p \nmid k'$. Thus, by Proposition 2.1, $\psi^n_S = \psi^k_S \circ \psi^{p^i}_S$ and $\psi^{n'}_S = \psi^{k'}_S \circ \psi^{p^i}_S$. By Lemma 5.1 (ii), we have $\psi^k_S = \psi^{k'}_S$. Hence $\psi^n_S(V) = \psi^{n'}_S(V)$, giving $D_n \approx D_{n'}$ by (5.6).

Now suppose that $n \geqslant q$. By (5.7), we have that $D_n \approx -\sum_{i=1}^{q-1} D_{n-i}B_i$ and $D_{n'} \approx -\sum_{i=1}^{q-1} D_{n'-i}B_i$. By induction on n we may assume that $D_{n-i} \approx D_{n'-i}$ for $i = 1, \ldots, q-1$. Hence $D_n \approx D_{n'}$, as required. □

Theorem 5.6. *Let G be a finite group and let K be a field of prime characteristic p. Then the Adams operations ψ^n_S on the Green ring R_{KG} are periodic in n if and only if the Sylow p-subgroups of G are cyclic.*

Proof. Suppose first that the Sylow p-subgroups of G are cyclic. In order to prove that ψ^n_S is periodic we may assume that K is algebraically closed, by [4, Lemma 2.4]. As in [2, p. 184], we say that a finite group is p-hypo-elementary if it is an extension of a p-group by a cyclic p'-group. Note that Proposition 5.5 shows that ψ^n_S is periodic in n on R_{KF} for every p-hypo-elementary subgroup F of G. Thus there exists a positive integer s such that ψ^n_S has period dividing s on R_{KF} for all such F. We show that $\psi^n_S(A) = \psi^{n+s}_S(A)$ for all $A \in R_{KG}$. By the choice of

s, the element $\psi_S^n(V) - \psi_S^{n+s}(V)$ of R_{KG} restricts to 0 in R_{KF} for every p-hypo-elementary subgroup F of G. Therefore, by [2, Corollary 5.6.9] (a result obtained from Conlon's induction theorem), we obtain $\psi_S^n(V) - \psi_S^{n+s}(V) = 0$, as required.

The converse was noted in the proof of Theorem 3.2. □

6 The Adams operations ψ_S^n for a cyclic p-group

In this section we consider the operations ψ_S^n on the Green ring R_{KC}, where K is a field of prime characteristic p and C is a cyclic p-group. As before we write $q = |C|$. When $q = 1$ we have $R_{KC} = \mathbb{Z}V_1$ so that ψ_S^n is the identity map for all $n > 0$. Thus from now on we assume that $q > 1$. We establish results for ψ_S^n analogous to those for ψ_Λ^n obtained in Section 4 and, using work of Symonds [8], we show that ψ_S^n may be expressed in terms of ψ_Λ^n.

If $\bar{K}$ is the algebraic closure of K then there is an isomorphism $R_{KC} \to R_{\bar{K}C}$ mapping V_r to $\bar{K} \otimes_K V_r$ for $r = 1, \ldots, q$. It is easy to see that this isomorphism commutes with the Heller maps and the Adams operations. Thus the results in Section 5 obtained under the assumption that K is algebraically closed hold, in the case $G = C$, without the need for algebraic closure. We shall use these results for arbitrary K without further comment.

Our first result is the analogue of parts (i) and (ii) of Theorem 4.8. We shall obtain the analogue of Theorem 4.8 (iii) in Corollary 6.3 below.

Theorem 6.1. *Let K be a field of prime characteristic p and let C be a cyclic p-group of order $q > 1$.*
(i) *The Adams operations ψ_S^n on R_{KC} are periodic in n, with minimum period σ, where $\sigma = 2q$ when p is odd and $\sigma = q$ when $p = 2$.*
(ii) *We have $\psi_S^n = \psi_S^{\sigma-n}$ for all $n < \sigma$.*

Proof. (i) By Proposition 5.5, ψ_S^n is periodic in n. Let σ be the minimum period. Then, by Proposition 4.6, we must have $q \mid \sigma$.

Suppose first that p is odd. By [5, Theorem 3.3] we have $\psi_S^n = \psi_S^{n+2p}$ on R_{KF} for all n such that $p \nmid n$ and for every subgroup F of C. Set $\pi = 2p$. Then π satisfies the conditions of Lemma 5.1 and hence, by Proposition 5.5, we have that $\psi_S^n = \psi_S^{n+2q}$ on R_{KC} for all $n > 0$. Thus $\sigma = q$ or $\sigma = 2q$. By the proof of Theorem 4.8 (i), we have $\psi_S^1 = \psi_\Lambda^1 \neq \psi_\Lambda^{q+1} = \psi_S^{q+1}$, and so $\sigma = 2q$.

Now suppose that $p = 2$. By [5, Corollary 3.5] we have $\psi_S^n = \psi_S^{n+2}$ on R_{KF} for all n such that $2 \nmid n$ and for every subgroup F of C. Set $\pi = 2$. Then π satisfies the conditions of Lemma 5.1 and hence, by Proposition 5.5, we have that $\psi_S^n = \psi_S^{n+q}$ on R_{KC} for all $n > 0$, giving $\sigma = q$.

(ii) Let p be arbitrary. By an argument entirely similar to the proof of Theorem 4.8 (ii), it can be shown that $\psi_S^n = \psi_S^{2q-n}$ for all $n < 2q$. However, $\psi_S^{2q-n} = \psi_S^{\sigma-n}$ for all $n < \sigma$, by (i). This gives the result. □

We use all the notation of Section 4 and write $\tilde{q} = q/p$. Thus $\tilde{q}$ is the order of $\tilde{C}$. Recall that $\Omega(V)$ denotes the Heller translate of a KC-module V and that Ω extends to a $\mathbb{Z}$-linear map $\Omega : R_{KC} \to R_{KC}$. It is easy to check that $\Omega(V_r) = V_{q-r}$, for $r = 1, \ldots, q$, where $V_0 = 0$. Hence $\Omega^2(V_r) \overset{\text{proj}}{=} V_r$ for $r = 1, \ldots, q$. Thus, working modulo projectives, we see that Ω^n is determined by the parity of n, with $\Omega^n(V_r) \overset{\text{proj}}{=} \Omega(V_r) \overset{\text{proj}}{=} V_{q-r}$ if n is odd and $\Omega^n(V_r) \overset{\text{proj}}{=} V_r$ if n is even. Similarly we have $\Omega : R_{K\tilde{C}} \to R_{K\tilde{C}}$ with $\Omega(\tilde{V}_r) = \tilde{V}_{\tilde{q}-r}$, for $r = 1, \ldots, \tilde{q}$.

Let $r \in \{1, \ldots, q\}$ and write $r = ap + b$ where $0 \leqslant b < p$. For $b > 0$ we have $q - r = p(\tilde{q} - (a+1)) + (p - b)$, where $0 < p - b < p$, while for $b = 0$ we have $q - r = p(\tilde{q} - a)$. Thus, by (4.1),

$$V_{q-r}{\downarrow_{\tilde{C}}} = (p-b)\tilde{V}_{\tilde{q}-a} + b\tilde{V}_{\tilde{q}-(a+1)}. \tag{6.1}$$

Recall that, for $A, B \in R_{KC}$, we have $A \overset{\text{proj}}{=} B$ if and only if $A - B \in \mathbb{Z}V_q$ while $A \overset{\text{ind}}{=} B$ if and only if $A - B \in \mathbb{Z}\{V_p, V_{2p}, \ldots, V_q\}$. We extend the notation to $\mathbb{Q}R_{KC}$ and $\mathbb{Q}R_{KC}[[t]]$ as follows. For $A, B \in \mathbb{Q}R_{KC}$ we write $A \overset{\text{proj}}{=} B$ when $A - B \in \mathbb{Q}V_q$ and $A \overset{\text{ind}}{=} B$ when $A - B \in \mathbb{Q}\{V_p, V_{2p}, \ldots, V_q\}$. For $A(t), B(t) \in \mathbb{Q}R_{KC}[[t]]$, where $A(t) = \sum_{i=0}^{\infty} A_i t^i$ and $B(t) = \sum_{i=0}^{\infty} B_i t^i$, we write $A(t) \overset{\text{proj}}{=} B(t)$ when $A_i \overset{\text{proj}}{=} B_i$ for all i and $A(t) \overset{\text{ind}}{=} B(t)$ when $A_i \overset{\text{ind}}{=} B_i$ for all i. Note that $A(t) \overset{\text{proj}}{=} B(t)$ implies that $A(t) \overset{\text{ind}}{=} B(t)$.

We extend the definition of Ω on R_{KC} to a $\mathbb{Q}$-linear map $\Omega : \mathbb{Q}R_{KC} \to \mathbb{Q}R_{KC}$. Let J be the subring of $\mathbb{Q}R_{KC}[[t]]$ consisting of all elements $\sum_{i=1}^{\infty} A_i t^i$ with zero constant term and let $\Omega^* : J \to J$ be defined by

$$\Omega^*\left(\sum_{i=1}^{\infty} A_i t^i\right) = \sum_{i=1}^{\infty} \Omega^i(A_i) t^i$$

for all $\sum_{i=1}^{\infty} A_i t^i \in J$. Clearly Ω^* is $\mathbb{Q}$-linear and it is easy to check, by (2.7), that $\Omega^*(XY) \overset{\text{proj}}{=} \Omega^*(X)\Omega^*(Y)$ for all $X, Y \in J$. Thus $\Omega^*(XY) \overset{\text{ind}}{=} \Omega^*(X)\Omega^*(Y)$ for all

$X, Y \in J$. It follows easily that, for all $X \in J$, we have

$$\Omega^*(\log(1+X)) \stackrel{\text{ind}}{=} \log(1+\Omega^*(X)). \tag{6.2}$$

We shall take $V = V_r$ in (5.5), where $r \in \{1, \ldots, q\}$. Note that, in the notation of (5.5), we must have $E = 1$ in R_{KC} because E is one-dimensional. Also, for all $A(t), B(t) \in \mathbb{Q}R_{KC}[[t]]$, we have $A(t) \approx B(t)$ if and only if $A(t) \stackrel{\text{ind}}{=} B(t)$, by the remark preceding Lemma 5.4. Thus (5.5) becomes

$$\psi_S(V_r, t) \stackrel{\text{ind}}{=} t\frac{d}{dt}\log(1 + S^1(V_r)t + \cdots + S^{q-1}(V_r)t^{q-1}) - t\frac{d}{dt}\log(1-t^q).$$

Now let $r \in \{\tilde{q}, \ldots, q\}$. By [8, Corollary 3.11] we have $S^n(V_r) \stackrel{\text{ind}}{=} \Omega^n(\Lambda^n(V_{q-r}))$ for all $n < q$. Thus

$$\psi_S(V_r, t) \stackrel{\text{ind}}{=} t\frac{d}{dt}\log\left(1 + \sum_{n=1}^{q-1}\Omega^n(\Lambda^n(V_{q-r}))t^n\right) + q\sum_{k=1}^{\infty}t^{kq}.$$

Since $\Lambda^n(V_{q-r}) = 0$ for $n > q-1$, we obtain, by the definition of Ω^*,

$$\psi_S(V_r, t) \stackrel{\text{ind}}{=} t\frac{d}{dt}\log\left(1 + \Omega^*\left(\sum_{n=1}^{\infty}\Lambda^n(V_{q-r})t^n\right)\right) + q\sum_{k=1}^{\infty}t^{kq}.$$

Thus, by (6.2) and the definition of $\Lambda(V_{q-r}, t)$,

$$\begin{aligned} \psi_S(V_r, t) &\stackrel{\text{ind}}{=} t\frac{d}{dt}\Omega^*(\log\Lambda(V_{q-r}, t)) + q\sum_{k=1}^{\infty}t^{kq} \\ &\stackrel{\text{ind}}{=} \Omega^*\left(t\frac{d}{dt}\log\Lambda(V_{q-r}, t)\right) + q\sum_{k=1}^{\infty}t^{kq}. \end{aligned}$$

Therefore, by (2.1) and the definition of Ω^*,

$$\psi_S(V_r, t) \stackrel{\text{ind}}{=} \sum_{n=1}^{\infty}(-1)^{n-1}\Omega^n(\psi_\Lambda^n(V_{q-r}))t^n + q\sum_{k=1}^{\infty}t^{kq}.$$

Comparing coefficients of t^n gives

$$\psi_S^n(V_r) \stackrel{\text{ind}}{=} \begin{cases} (-1)^{n-1}\Omega^n(\psi_\Lambda^n(V_{q-r})) & \text{if } q \nmid n, \\ (-1)^{n-1}\Omega^n(\psi_\Lambda^n(V_{q-r})) + qV_1 & \text{if } q \mid n, \end{cases} \tag{6.3}$$

for all $n > 0$ and all r such that $\tilde{q} \leqslant r \leqslant q$.

Theorem 6.2. *Let K be a field of prime characteristic p and let C be a cyclic p-group of order $q > 1$. The Adams operations ψ^n_Λ and ψ^n_S on R_{KC} satisfy*

$$\psi^n_S(V_r) \overset{\text{proj}}{=} (-1)^{n-1}\Omega^n(\psi^n_\Lambda(V_{q-r})) + (n,q)V_{q/(n,q)}$$

for all r such that $q/p \leqslant r \leqslant q$ and all $n > 0$.

Theorem 6.2 yields

$$\psi^n_S(V_r) = (-1)^{n-1}\Omega^n(\psi^n_\Lambda(V_{q-r})) + (n,q)V_{q/(n,q)} + aV_q,$$

for some $a \in \mathbb{Z}$. If $\psi^n_\Lambda(V_{q-r})$ is known then we may calculate $s = \delta(\Omega^n(\psi^n_\Lambda(V_{q-r})))$. Since $\delta(\psi^n_S(V_r)) = r$, by (2.3), we find that $a = (r + (-1)^n s - q)/q$. Hence Theorem 6.2 determines $\psi^n_S(V_r)$ completely in terms of $\psi^n_\Lambda(V_{q-r})$, for $r \in \{q/p, \ldots, q\}$. Suppose now that $r \in \{1, \ldots, q\}$, and choose j such that $1 \leqslant p^{j-1} \leqslant r \leqslant p^j \leqslant q$. Then C has a factor group $C(p^j)$ of order p^j, and $R_{KC(p^j)}$ may be regarded as the subring of R_{KC} spanned by $V_1, V_2, \ldots, V_{p^j}$ (see the remarks preceding (4.3)). By Theorem 6.2 applied to $C(p^j)$ we may express $\psi^n_S(V_r)$ in terms of $\psi^n_\Lambda(V_{p^j-r})$. Thus Theorem 6.2 enables ψ^n_S to be completely determined from ψ^n_Λ.

Proof of Theorem 6.2. As before, write $\tilde{q} = q/p$. We prove the result by induction on q. First suppose that $q = p$ and let $r \in \{1, \ldots, p\}$. By (6.3) we have

$$\psi^n_S(V_r) \overset{\text{ind}}{=} (-1)^{n-1}\Omega^n(\psi^n_\Lambda(V_{p-r})) + (n,p)V_{p/(n,p)},$$

since V_p is induced. The result then follows from the fact that all induced KC-modules are projective, since $q = p$.

Now let $q > p$ and assume that the result holds for $\tilde{C}$. Let $r \in \{\tilde{q}, \ldots, q\}$ and set

$$Y = \psi^n_S(V_r) + (-1)^n\Omega^n(\psi^n_\Lambda(V_{q-r})).$$

By (2.6) and (2.8), we have

$$Y{\downarrow_{\tilde{C}}} \overset{\text{proj}}{=} \psi^n_S(V_r{\downarrow_{\tilde{C}}}) + (-1)^n\Omega^n(\psi^n_\Lambda(V_{q-r}{\downarrow_{\tilde{C}}})).$$

Write $r = ap + b$ where $0 \leqslant b < p$. Then, by (4.1) and (6.1), we have

$$\begin{aligned} Y{\downarrow_{\tilde{C}}} \overset{\text{proj}}{=}\ & (p-b)\left[\psi^n_S(\tilde{V}_a) + (-1)^n\Omega^n(\psi^n_\Lambda(\tilde{V}_{\tilde{q}-a}))\right] \\ & + b\left[\psi^n_S(\tilde{V}_{a+1}) + (-1)^n\Omega^n(\psi^n_\Lambda(\tilde{V}_{\tilde{q}-(a+1)}))\right]. \end{aligned}$$

Note that $\tilde{q}/p \leqslant a \leqslant \tilde{q}$ since $\tilde{q} \leqslant r \leqslant q$. Also, if $b > 0$ then $\tilde{q}/p < a+1 \leqslant \tilde{q}$. Hence, by the inductive hypothesis, we may replace each of the expressions contained in square brackets above by $(n, \tilde{q})\tilde{V}_{\tilde{q}/(n,\tilde{q})}$. Thus

$$Y{\downarrow_{\tilde{C}}} \overset{\text{proj}}{=} p(n, \tilde{q})\tilde{V}_{\tilde{q}/(n,\tilde{q})}. \tag{6.4}$$

Consider the case where $q \nmid n$, so that $(n, q) = (n, \tilde{q}) < q$. Then, by (6.3), Y is induced. Hence $Y - (n, q)V_{q/(n,q)}$ is induced. Also, by (6.4),

$$(Y - (n, q)V_{q/(n,q)}){\downarrow_{\tilde{C}}} \overset{\text{proj}}{=} p(n, \tilde{q})\tilde{V}_{\tilde{q}/(n,\tilde{q})} - p(n, q)\tilde{V}_{\tilde{q}/(n,q)} = 0.$$

Therefore $Y - (n, q)V_{q/(n,q)} \overset{\text{proj}}{=} 0$ by Lemma 4.1 (ii). This gives the required result.

Finally suppose that $q \mid n$, so that $(n, q) = q$ and $(n, \tilde{q}) = \tilde{q}$. Then, by (6.3), $Y - qV_1$ is induced. Also, by (6.4),

$$(Y - qV_1){\downarrow_{\tilde{C}}} \overset{\text{proj}}{=} p\tilde{q}\tilde{V}_1 - q\tilde{V}_1 = 0.$$

Therefore $Y - qV_1 \overset{\text{proj}}{=} 0$ by Lemma 4.1 (ii). This gives the required result. □

As before δ denotes the endomorphism of R_{KC} satisfying $\delta(V_r) = rV_1$ for all r.

Corollary 6.3. *In the notation of Theorem* 6.1*, where σ denotes the minimum period of ψ^n_S, we have $\psi^\sigma_S = \delta$.*

Proof. Since $\psi^\sigma_S = \psi^{2q}_S$, it is sufficient to show that $\psi^{2q}_S(V_r) = rV_1$ for $r = 1, \ldots, q$. Suppose first that $r \in \{\tilde{q}, \ldots, q\}$. Then, by Theorem 6.2, we have

$$\psi^{2q}_S(V_r) \overset{\text{proj}}{=} -\psi^{2q}_\Lambda(V_{q-r}) + qV_1.$$

Thus, by Theorem 4.8 (iii), we have $\psi^{2q}_S(V_r) \overset{\text{proj}}{=} -(q - r)V_1 + qV_1 = rV_1$. Since $\delta(\psi^{2q}_S(V_r)) = r$ it follows that $\psi^{2q}_S(V_r) = rV_1$.

Now suppose that $r \leqslant \tilde{q}$. Then we may regard V_r as a module for the factor group $C(\tilde{q})$ of C with order $\tilde{q}$ and, by induction, we may assume that $\psi^{2\tilde{q}}_S(V_r) = rV_1$. However, for $R_{KC(\tilde{q})}$, the minimum period of ψ^n_S divides $2\tilde{q}$. Hence $\psi^{2q}_S(V_r) = rV_1$, as required. □

7 Minimum periods

Let G be a finite group and K a field of characteristic $p \geqslant 0$. Throughout this section we shall assume that the Adams operations ψ_Λ^n and ψ_S^n are periodic in n. Thus either $p = 0$ or p is a prime and the Sylow p-subgroups of G are cyclic. We shall give lower bounds for the minimum periods of ψ_Λ^n and ψ_S^n.

Let $G_{p'}$ denote the set of all elements of G of order prime to p, let e denote the exponent of G, and let e' denote the least common multiple of the orders of the elements of $G_{p'}$. Also, write λ for the minimum period of ψ_Λ^n and σ for the minimum period of ψ_S^n. We begin by showing that λ and σ are divisible by e'.

Let V be the regular KG-module and let χ denote the Brauer character of V. Thus, for all $g \in G_{p'}$, we have $\chi(g) = |G|$ if $g = 1$ and $\chi(g) = 0$ otherwise. We extend the definition of Brauer character to elements of R_{KG} by linearity (as in the proof of Lemma 3.1) and, for each $n > 0$, we write χ_Λ^n for the Brauer character of $\psi_\Lambda^n(V)$ and χ_S^n for the Brauer character of $\psi_S^n(V)$. By [4, Lemma 2.6], $\chi_\Lambda^n(g) = \chi(g^n) = \chi_S^n(g)$ for all $g \in G_{p'}$. Since $\psi_\Lambda^{e'} = \psi_\Lambda^{e'+\lambda}$ and $\psi_S^{e'} = \psi_S^{e'+\sigma}$ it follows that $\chi(g^\lambda) = \chi(1) = \chi(g^\sigma)$ for all $g \in G_{p'}$. Thus $g^\lambda = g^\sigma = 1$ for all $g \in G_{p'}$, giving that $e' \mid \lambda$ and $e' \mid \sigma$.

If $|G|$ is not divisible by p then $G_{p'} = G$ and $e' = e$. Thus, by the argument above and Lemma 3.1, we have the following result.

Proposition 7.1. *Let G be a finite group and K a field such that $|G|$ is not divisible by the characteristic of K. Then ψ_Λ^n and ψ_S^n are both periodic in n with minimum period e, where e is the exponent of G.*

From now on we shall assume that K has prime characteristic p.

Proposition 7.2. *Let K be a field of prime characteristic p and let G be a finite group with a cyclic Sylow p-subgroup C of order $q > 1$. Then ψ_Λ^n and ψ_S^n are periodic in n with minimum periods divisible by the least common multiple of 2 and the exponent of G.*

Proof. Let λ, σ, e and e' be as before and let V be the regular KG-module. Then $\psi_\Lambda^n(V{\downarrow_C}) = \psi_\Lambda^{n+\lambda}(V{\downarrow_C})$ and $\psi_S^n(V{\downarrow_C}) = \psi_S^{n+\sigma}(V{\downarrow_C})$ for all $n > 0$. Since $V{\downarrow_C}$ is a free KC-module, it follows from Propositions 4.5 and 4.6 that $q \mid \lambda$ and $q \mid \sigma$. But, as we have seen, $e' \mid \lambda$ and $e' \mid \sigma$. Hence $e \mid \lambda$ and $e \mid \sigma$. This completes the proof for $p = 2$. Thus we now assume that p is odd. Consider the KC-module V_2

and let $M = V_2{\uparrow}^G$. It is straightforward to prove from Mackey's decomposition theorem (see a similar argument in Section 3) that $M{\downarrow}_C = rV_2 + W$, where r is a positive integer and W is a sum of regular modules for factor groups of C regarded as KC-modules. For all $n > 0$ we have

$$\psi^n_\Lambda(rV_2 + W) = \psi^{n+\lambda}_\Lambda(rV_2 + W), \quad \psi^n_S(rV_2 + W) = \psi^{n+\sigma}_S(rV_2 + W).$$

However, by Lemma 2.2, $\psi^n_\Lambda(W) = \psi^n_S(W) = W$ for all n such that $p \nmid n$. Thus $\psi^1_\Lambda(V_2) = \psi^{\lambda+1}_\Lambda(V_2)$ and $\psi^1_S(V_2) = \psi^{\sigma+1}_S(V_2)$. By [5, Theorem 5.1] it now follows that λ and σ are divisible by 2. □

The case $q = p$ was considered in [4, Theorem 7.2] and it was shown that $\lambda \mid 2e$ and $\sigma \mid 2e$, where e is the exponent of G. Thus, in view of the results above and our results for the case $G = C$, we make the following conjecture.

Conjecture. Let K be a field of prime characteristic p and let G be a finite group with cyclic Sylow p-subgroups. Let e be the exponent of G. Then the minimum period of ψ^n_Λ is e or $2e$, and the minimum period of ψ^n_S is e or $2e$.

Acknowledgements. Work supported by EPSRC Standard Research Grant EP/G024898/1. We are indebted to Peter Symonds for suggesting the use of Conlon's induction theorem in Section 5.

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